Scale decomposition in compressible turbulence

Hussein Aluie

Applied Mathematics and Plasma Physics (T-5) & Center for Non-linear Studies,

Los Alamos National Laboratory, MS-B258 Los Alamos, NM 87545, USA

Abstract

This work presents a rigorous framework based on coarse-graining to analyze highly compressible turbulence. We show how the sole requirement that viscous effects on the dynamics of large-scale momentum and kinetic energy be negligible —an inviscid criterion— naturally leads to a density weighted coarse-graining of the velocity field. Such a coarse-graining method is already known in the literature as Favre filtering; however its use has been primarily motivated by appealing modelling properties rather than underlying physical considerations. We also prove that kinetic energy injection can be localized to the largest scales by proper stirring, and discuss the special case of buoyancy-driven flows subject to an external spatially-uniform acceleration field. We conclude that a range of scales can exist over which the mean kinetic energy budget is dominated by inertial processes and is immune from contributions due to molecular viscosity and external stirring.

Key Words: compressible turbulence, nonlinear scale analysis, scale decomposition

1 Introduction

This paper is the first in a series which studies the physical nature of compressible turbulence. The aim here is to give a systematic, theoretical approach based on coarse-graining (or filtering) to analyze non-linear scale interactions in such flows. It builds upon previous work of [10] and [3, 4]. There are several motivations for this work. First, there is no unique way to specifying a notion of scale, such as defining large-scale momentum and large-scale kinetic energy, in compressible turbulence. The traditional approach in this subject has employed density-weighted averaging, also known as Favre averaging, to decompose a flow into large-scale and turbulent components. Using Favre averaging, density, $\rho(\mathbf{x})$, and velocity, $\mathbf{u}(\mathbf{x})$, are combined to yield a large-scale momentum, $\langle\rho\mathbf{u}\rangle$, and a large-scale kinetic energy, $\frac{1}{2}\langle\rho\mathbf{u}\rangle^2/\langle\rho\rangle$, with $\langle\dots\rangle$ denoting a space-average $\frac{1}{V}\int_V d\mathbf{x}(\dots)$. However, such a decomposition has been borne out of convenience to modelers and practitioners rather than physical considerations. It seems that *a priori* there is no fundamental reason to favor Favre averaging over any other combination of ρ and $\mathbf{u}$, from an infinite number of possibilities.

Second, and more importantly, this paper provides the foundation for subsequent work in which we attempt to answer basic questions about the physics of compressible turbulence. While the classical ideas of Richardson, Kolmogorov, and Onsager form the cornerstone for our modern understanding of incompressible turbulence, there are no grounds for extending such a theory to compressible flows. The potent ideas of an inertial range and universality are often invoked without physical basis in compressible turbulence. Elementary questions on the possible existence of a scale-range which is immune from direct effects of viscosity and large scale forcing, on whether energy is transferred to small scales through a cascade process, and whether such a cascade is local in scale, all remain unanswered. Resolving these questions is necessary to warrant the concept of an inertial range and to justify the existence of universal statistics of turbulent fluctuations. Furthermore, Kolmogorov's 4/5-th law for the energy flux is an exact result which has no counterpart in compressible turbulence. An analogous result would be essential for attempting to predict the scaling of spectra and structure functions.

This work is also very useful from the standpoint of numerical modelling. Compressible flows, especially in astrophysical systems, often involve a huge range of scales which cannot be simulated directly. The coarse-graining approach provides a theoretical basis for constructing models of turbulence that may faithfully reflect the dynamics at unresolved scales. The formalism that we employ is the same as that used in large-eddy simulation (LES) modelling of turbulent flows. This work thus provides a theoretical complement to those modelling efforts. However, while the equations we analyze coincide (to a considerable extent) with those that are employed in LES of compressible turbulence, their use here will be for rather different purposes. In LES, plausible but uncontrolled closures are adopted for the subscale terms, whereas the aim here and in subsequent papers, following [3, 4], is to develop several

exact estimates and some general physical understanding of these terms. Another difference is that LES generally takes the scale parameter ℓ to be a fixed length of the order of the "integral scale" L. Our interest here is rather to probe *all scales in the flow*, including limits of small $\ell \ll L$.

In this paper, we shall show how a decomposition based on Favre filtering (and averaging) comes out naturally from the physical requirement that viscous effects have a negligible role in the dynamics of large-scale momentum and large-scale kinetic energy. We call this the *inviscid criterion*. Using the coarse-graining approach, we will prove rigorously the existence of an intermediate range of scales over which viscous dissipation and external kinetic energy injection can be made to vanish. Our decomposition also leads to two terms responsible for transferring kinetic energy across scales and constitute the so-called "subgrid scale flux," which we shall discuss in detail.

The outline of this paper is as follows. In § 2 we present preliminary definitions and discussion. In § 3 we show how viscous dynamics can be isolated to the smallest scales by a proper scale-decomposition and in § 4 we discuss the inertial dynamics based on such a decomposition. In § 5 we prove that it is possible to localize kinetic energy injection to the largest scales in a system and discuss buoyancy-driven (Rayleigh-Taylor) flows as a special case. In § 6, we examine contributions from compressibility effects to the flux of kinetic energy across scales. We conclude with § 7 and two appendices, A and B, which contain detailed proofs of our results.

2 Preliminaries

2.1 Governing dynamics

In this paper we study the dynamics at various scales through a direct analysis of the compressible Navier Stokes equations without the use of any closure approximation. The equations are those of continuity (1), momentum (2), and either internal energy (3) or total energy (4):

$$\partial_t \rho \quad +\partial_j(\rho u_j) = 0 \tag{1}$$

$$\partial_t(\rho u_i) \quad +\partial_j(\rho u_i u_j) = -\partial_i P + \mu\partial_j(\partial_j u_i + \frac{1}{3}\partial_m u_m \delta_{ij}) + \rho F_i \tag{2}$$

$$\partial_t(\rho e) \quad +\partial_j\left\{\rho e u_j - \mu(u_m\partial_m u_j - u_j\partial_m u_m)\right\} = -P\partial_j u_j + \mu|\partial_j u_i|^2 + \frac{\mu}{3}|\partial_j u_j|^2 - \partial_j q_j \tag{3}$$

$$\partial_t(\rho E) \quad +\partial_j(\rho E u_j) = -\partial_j(P u_j) + \mu\partial_j\{u_i[(\partial_j u_i + \partial_i u_j) - \frac{2}{3}\partial_m u_m \delta_{ij}]\} - \partial_j q_j + \rho u_i F_i \tag{4}$$

Here, $\mathbf{u}$ is velocity, ρ is density, e is internal energy per unit mass, $E = |\mathbf{u}|^2/2 + e$ is total energy per unit mass, P is pressure, μ is dynamic viscosity, $\mathbf{F}$ is an external acceleration field stirring the fluid, $\mathbf{q} = -\kappa\boldsymbol{\nabla}T$ is the heat flux with a conduction coefficient κ and temperature T. For convenience, we have assumed a zero bulk viscosity even though all our analysis applies to the more general case. We

have assumed that $\mu = \nu\rho$ is independent of $\mathbf{x}$. In general, $\mu(\mathbf{x}) \sim T(\mathbf{x})^{\alpha}$, hence our assumption should hold in isothermal flows. It is also reasonable in flows which are turbulent enough such that temperature conduction is enhanced, making μ a weak function of $\mathbf{x}$.

2.2 Coarse-graining

We first present a general approach to analyzing scale interactions in a turbulent flow. Following [10] and [4], we use a simple filtering technique common in the LES literature to resolve turbulent fields simultaneously in scale and in space.

For any field $\mathbf{a}(\mathbf{x})$, a "coarse-grained" or (low-pass) filtered field, which contains modes at scales $> \ell$, is defined in d-dimensions as

$$\overline{\mathbf{a}}_{\ell}(\mathbf{x}) = \int d^d\mathbf{r}\ G_{\ell}(\mathbf{r})\mathbf{a}(\mathbf{x}+\mathbf{r}), \tag{5}$$

where $G(\mathbf{r})$ is a convolution kernel. It can be any real-valued function which is sufficiently smooth, decays sufficiently rapidly for large r, and is normalized so that $\int d^d\mathbf{r}\ G(\mathbf{r}) = 1$. It is assumed furthermore that G is centered, $\int d^d\mathbf{r}\ \mathbf{r}G(\mathbf{r}) = \mathbf{0}$, and with the main support in a ball of unit radius, $\int d^d\mathbf{r}\ |\mathbf{r}|^2 G(\mathbf{r}) = \mathcal{O}(1)$. Its dilation in d-dimensions, $G_{\ell}(\mathbf{r}) \equiv \ell^{-d}G(\mathbf{r}/\ell)$, will share these properties except that its main support will be in a ball of radius ℓ. If $G(\mathbf{r})$ is also non-negative, then (5) may be interpreted as a local space average. Note that $G(\mathbf{r})$ can be chosen so that both it and its Fourier transform $\hat{G}(\mathbf{k})$ are positive and infinitely differentiable, with $\hat{G}(\mathbf{k})$ also compactly supported inside a ball of radius 1 about the origin in Fourier space and with $G(\mathbf{r})$ decaying faster than any power r^{-p} as $r \to \infty$. See for instance Appendix A in [6] for explicit examples. It can be shown that for any kernel $G(\mathbf{r})$ with the above properties, a coarse-grained function $\overline{f}_{\ell}(\mathbf{x})$ is infinitely differentiable [1].

We can also define a complimentary high-pass filter which retains only modes at scales $< \ell$ by

$$\mathbf{a}_{\ell}^{'}(\mathbf{x}) = \mathbf{a}(\mathbf{x}) - \overline{\mathbf{a}}_{\ell}(\mathbf{x}). \tag{6}$$

In the rest of our paper, we shall take the liberty of dropping subscript ℓ whenever there is no risk of ambiguity.

It has been remarked by [17] and []Garnieretal09 that filtering a strong discontinuity in a field, such as an external shock from an explosion, introduces "parasitic" contributions which can overwhelm the turbulent fluctuations at small scales. Since our primary purpose in this work is a fundamental physical understanding, rather than modelling, of non-linear scale interactions, and since such a strong shock would interact with the flow, we consider it only natural to include its contributions to sub-scales $< \ell$.

The filtering operation (5) is linear and commutes with space (and time) derivatives. We can apply it to the continuity and momentum equations (1)-(2) to describe dynamics of large-scale fields. However,

[1]Under the very weak requirement that $\int_{\Omega} d\mathbf{x}|f(\mathbf{x})| < \infty$ over the domain Ω of the flow.

there is no unique way to filter these equations. For example, we may define a large-scale momentum field either as $\overline{\rho}_\ell \overline{\mathbf{u}}_\ell$ or as $\overline{\rho \mathbf{u}}_\ell$. Similarly, a large-scale kinetic energy may be defined as $\frac{1}{2}\overline{\rho}_\ell |\overline{\mathbf{u}}_\ell|^2$ or $\frac{1}{2}|\overline{\sqrt{\rho}\mathbf{u}}_\ell|^2$.

3 Identifying the viscous range

A key idea of this paper is that the scale-decomposition of momentum and kinetic energy should satisfy the *inviscid criterion*, *i.e.* it should guarantee that viscous contributions are negligible at large enough length-scales. This is necessary for the study of inertial range dynamics if such a scale-range exists in compressible turbulence.

3.1 Scale decomposition

Under the assumption that $\mu(\mathbf{x}) = \mu$ is independent of spatial position $\mathbf{x}$, we can coarse-grain eq.(2) and commute the filter with space derivatives in the viscous diffusion terms. We obtain

$$\partial_t \overline{\rho u}_i + \partial_j (\overline{\rho u_i u_j}) = -\partial_i \overline{P} + \mu \partial_j \{[(\partial_j \overline{u}_i + \partial_i \overline{u}_j) - \frac{2}{3}\partial_m \overline{u}_m \delta_{ij}]\} + \overline{\rho F}_i. \tag{7}$$

With such a decomposition, the functional form of viscous terms in (7) is similar to their counterpart in incompressible flows. If $\int d\mathbf{x}|\mathbf{u}(\mathbf{x})|^2 < \infty$, it can be shown rigorously that each of the viscous terms in eq. (7) is bounded by (const.)$\mu\ell^{-2}$ at every point $\mathbf{x}$. Therefore, the large-scale momentum, defined as $\overline{\rho \mathbf{u}}_\ell$, does not diffuse under the action of molecular viscosity when $\mu/\ell^2 \ll 1$. The proof is standard in real analysis and for applications in turbulence theory, see [5] and [1]. We repeat the proof in Appendix A for completeness. The idea behind it is simple; a term $\mu\nabla^2\overline{\mathbf{u}}$ may be expressed in terms of a big-O bound, $O(\mu\,\delta u(\ell)/\ell^2)$, which becomes negligible as $\mu \to 0$. Here, an increment is $\delta\mathbf{u}(\boldsymbol{\ell}) = u(\mathbf{x}+\boldsymbol{\ell}) - u(\mathbf{x})$.

The reader might question the value of a careful proof when one can arrive at the same conclusion by a simple dimensional argument. To illustrate the potential pitfalls of dimensional reasoning here, consider the quantity $\mu\overline{\boldsymbol{\nabla}\mathbf{u} : \boldsymbol{\nabla}\mathbf{u}}_\ell$. It may be argued that this should also vanish as $\mu \to 0$ for some fixed $\ell > 0$. However, it is well known in turbulence literature that it does not (see for example [19, 20, 16]). The problem lies in that $\mu\overline{\boldsymbol{\nabla}\mathbf{u} : \boldsymbol{\nabla}\mathbf{u}}_\ell$ cannot be expressed in terms of quantities at scale ℓ, such as $O(\mu\,\delta u(\ell)^2/\ell^2)$, as one might innocently expect. When $\mu \to 0$ gradients can become unbounded and the term $\overline{\boldsymbol{\nabla}\mathbf{u} : \boldsymbol{\nabla}\mathbf{u}}_\ell$ diverges. Even though $\overline{\boldsymbol{\nabla}\mathbf{u} : \boldsymbol{\nabla}\mathbf{u}}_\ell$ has small wavenumber modes $< K \sim \ell^{-1}$, it can be dominated by contributions from $\hat{\mathbf{u}}(\mathbf{q})$ with wavenumbers $Q \gg K$ due to the convolution $\boldsymbol{\nabla}\mathbf{u} : \boldsymbol{\nabla}\mathbf{u}$ in Fourier space. Put more explicitly, while the product $\widehat{\boldsymbol{\nabla}\mathbf{u}}(\mathbf{q}) : \widehat{\boldsymbol{\nabla}\mathbf{u}}(\mathbf{k}-\mathbf{q})e^{i\mathbf{q}\cdot\mathbf{x}}e^{i(\mathbf{k}-\mathbf{q})\cdot\mathbf{x}}$ has a Fourier mode at wavenumber $|\mathbf{k}| < K$, it is proportional to $\sim Q^2$. This example has a direct bearing on our definition of large-scale momentum. If we were to define large-scale momentum as $\overline{\rho}_\ell\overline{\mathbf{u}}_\ell$ rather than $\overline{\rho\mathbf{u}}_\ell$ as we did above, a viscous term in the balance equation would have the form $\mu\overline{\rho}_\ell\overline{\rho^{-1}\nabla^2\mathbf{u}}_\ell$. Here,

again, the filtering operation would not commute with the laplacian and, due to possibly dominant contributions from high wavenumber modes $\gg \ell^{-1}$, we would not be able to guarantee *a priori* that viscous terms are negligible at large ℓ.

In the analysis above we needed to assume that μ is a constant. The results should still hold if $\mu(\mathbf{x})$ only varies over large length-scales $\gg \ell$ or if its fluctuations are small relative to the mean $|\mu(\mathbf{x}) - \langle\mu\rangle| \ll \langle\mu\rangle$. *A priori* tests of compressible turbulence by []Vremanetal95, []VremanThesis, and []Martinetal00 seem to suggest that, indeed, the additional non-linearity in viscous terms introduced by Sutherland's law is small.

The scale decompostion employed in the large-scale momentum balance (7) is equivalent to traditional Favre filtering (see for example [9]), where a Favre filtered function is weighted by the density:

$$\widetilde{f}_\ell(\mathbf{x}) \equiv \frac{\overline{\rho f}_\ell(\mathbf{x})}{\overline{\rho}_\ell(\mathbf{x})}. \tag{8}$$

The operator $\widetilde{(\cdot)}$ is linear but does not commute with derivatives. The large-scale momentum balance (7) can be rewritten using definition (8) as

$$\begin{aligned} \partial_t\overline{\rho}\widetilde{u}_i + \partial_j(\overline{\rho}\widetilde{u}_i\ \widetilde{u}_j) &= -\partial_j\left(\overline{\rho}\ \widetilde{\tau}(u_i,u_j)\right) - \partial_i\overline{P} \\ &+ \mu\partial_j\{[(\partial_j\overline{u}_i + \partial_i\overline{u}_j) - \frac{2}{3}\partial_m\overline{u}_m\delta_{ij}]\} + \overline{\rho}\widetilde{F}_i. \end{aligned} \tag{9}$$

This is the same as the "bare" momentum equation (2) itself but with an additional contribution from *turbulent stress*,

$$\overline{\rho}\widetilde{\tau}(u_i,u_j) \equiv \overline{\rho}(\widetilde{u_i u_j} - \widetilde{u}_i\ \widetilde{u}_i), \tag{10}$$

which accounts for the effect of eliminated scales $< \ell$ and vanishes identically in the absence of fluctuations at those small scales. One can also obtain a continuity equation for large-scale density:

$$\partial_t\overline{\rho} + \partial_i(\overline{\rho}\widetilde{u}_i) = 0. \tag{11}$$

A main advantage of the filtering approach to analyzing turbulent flows is an ability to resolve the relevant physical processes both *in scale* and *in space* as is apparent from the balance eqs. (9),(11). They describe the evolution of large-scale momentum and large-scale density at every $\mathbf{x}$ in the flow and at variable resolution ℓ. Using $\overline{\rho}_\ell$ and $\overline{\rho\mathbf{u}}_\ell$, and eqs. (9),(11), it is also straightforward to derive a simple dynamical equation for the kinetic energy density at scales $> \ell$, for arbitrary ℓ. This yields

$$\partial_t\overline{\rho}_\ell\frac{|\widetilde{\mathbf{u}}_\ell|^2}{2} + \boldsymbol{\nabla}\boldsymbol{\cdot}\mathbf{J}_\ell = -\Pi_\ell - \Lambda_\ell + \overline{P}_\ell\boldsymbol{\nabla}\boldsymbol{\cdot}\overline{\mathbf{u}}_\ell - D_\ell + \epsilon_\ell^{inj}, \tag{12}$$

where $\mathbf{J}_\ell(\mathbf{x})$ is space transport of large-scale kinetic energy, $\Pi_\ell(\mathbf{x}) + \Lambda_\ell(\mathbf{x})$, which we examine closely in § 4, is usually called the *subgrid scale (SGS) kinetic energy flux* to scales $< \ell$, $-\overline{P}_\ell\boldsymbol{\nabla}\boldsymbol{\cdot}\overline{\mathbf{u}}_\ell$ is large-scale

pressure dilatation, $D_\ell(\mathbf{x})$ is viscous dissipation acting on scales $> \ell$, and $\epsilon_\ell^{inj}(\mathbf{x})$ is the energy injected due to external stirring. These terms are defined as

$$\Pi_\ell(\mathbf{x}) = -\overline{\rho}\,\partial_j\widetilde{u}_i\,\widetilde{\tau}(u_i,u_j) \tag{13}$$

$$\Lambda_\ell(\mathbf{x}) = \frac{1}{\overline{\rho}}\partial_j\overline{P}\,\overline{\tau}(\rho,u_j) \tag{14}$$

$$D_\ell(\mathbf{x}) = \mu\left[\partial_j\widetilde{u}_i\,\partial_j\overline{u}_i + \frac{1}{3}\partial_i\widetilde{u}_i\,\partial_j\overline{u}_j\right] \tag{15}$$

$$J_j(\mathbf{x}) = \overline{\rho}\frac{|\widetilde{\mathbf{u}}|^2}{2}\widetilde{u}_j + \overline{P}\overline{u}_j + \widetilde{u}_i\overline{\rho}\widetilde{\tau}(u_i,u_j) - \mu\widetilde{u}_i\partial_j\overline{u}_i - \frac{\mu}{3}\widetilde{u}_j\partial_i\overline{u}_i \tag{16}$$

$$\epsilon_\ell^{inj}(\mathbf{x}) = \widetilde{u}_i\,\overline{\rho}\widetilde{F}_i \tag{17}$$

where we employed in (14) the notation

$$\overline{\tau}_\ell(f,g) \equiv \overline{(fg)_\ell} - \overline{f}_\ell\overline{g}_\ell \tag{18}$$

for 2^{nd}-order *generalized central moments* of any fields $f(\mathbf{x}), g(\mathbf{x})$ (see [10]).

Just as we have shown that viscous diffusion of large-scale momentum $\overline{\rho\mathbf{u}} = \overline{\rho}\widetilde{\mathbf{u}}$ is negligible, we can also rigorously prove under very weak conditions that viscous dissipation $D_\ell(\mathbf{x})$ of large-scale kinetic energy $\frac{1}{2}\overline{\rho}|\widetilde{\mathbf{u}}|^2$ vanishes at every point $\mathbf{x}$ when $\mu/\ell^2 \ll 1$. The proof, given in Appendix A, uses standard tools from real analysis.

3.2 The inviscid criterion

In one of his original articles, [7] motivated the usage of density-weighted averaging by the fact that average mass of a fluid in a volume V advected by the large-scale velocity $\langle\rho\mathbf{u}\rangle/\langle\rho\rangle$ is conserved. We shall now briefly repeat Favre's argument. The change of average mass in a volume V advected with some large-scale velocity $\mathbf{u}^*$ is

$$\int_V d^3\mathbf{x}\;\partial_t\langle\rho\rangle + \boldsymbol{\nabla\cdot}(\langle\rho\rangle\mathbf{u}^*) = \int_V d^3\mathbf{x}\;\boldsymbol{\nabla\cdot}(\langle\rho\rangle\mathbf{u}^* - \langle\rho\mathbf{u}\rangle)\,, \tag{19}$$

where now $\langle\dots\rangle$ denotes ensemble averaging. The equality follows from using the ensemble averaged continuity eq. (1). Favre averaged velocity $\widetilde{\mathbf{u}}$ is defined as the choice of $\mathbf{u}^*$ which makes average flux of mass due to fluctuations (or turbulence) vanish, $\langle\rho\rangle\widetilde{\mathbf{u}} - \langle\rho\mathbf{u}\rangle := 0$. The same argument carries over to spatially filtered dynamics, where now $\widetilde{\mathbf{u}}_\ell$ is defined as the choice of large-scale velocity which does not lead to subgrid terms in the filtered continuity eq. (11).

There are two comments we would like to make concerning Favre's argument. First, the special property large-scale velocity $\widetilde{\mathbf{u}}$ enjoys, *i.e.* suppressing turbulent diffusion of mass in eq. (11), does not logically imply by itself that a density-weighted decomposition is a necessary choice in the mass balance (11). It is certainly not unphysical for a turbulent flow to diffuse mass and, in this respect, the choice of a large-scale velocity field would depend on the particular purpose of an investigation. Second, a criterion requiring that turbulent mass diffusion be zero has no logical implication on the scale decomposition of momentum and kinetic energy. Unlike in eq. (11), the Favre decomposition results in turbulent diffusion and dissipation of large-scale momentum and kinetic energy as seen from eqs. (9),(12).

Yet, density-weighted filtering is used extensively in LES of compressible turbulence due to its modelling appeal. One of the perceived advantages is the absence of subgrid terms to be modelled in the coarse-grained continuity eq. (11). Another reason is that Favre filtered equations (9),(12) are structurally similar to their classically filtered counterparts in incompressible flows, which allows for carrying over models from the incompressible LES literature. Furthermore, none of the subgrid scale terms is a function of pressure which practitioners try to avoid modelling. When using the ideal gas law, $P = (\text{const.})\rho T$, as the equation of state, there is also an added advantage that filtered pressure, $\overline{P}$, can be expressed as a function of resolved quantities, $\overline{\rho}$ and $\widetilde{T}$, without additional subgrid terms.

What we have shown above is that Favre decomposition of momentum and kinetic energy satisfies the *inviscid criterion*. It guarantees that viscous contributions are negligible at large enough length-scales. Such a decomposition of momentum and kinetic energy is borne out of a physical requirement, irrespective of practical modelling considerations. We remark, however, that it may not be the unique decomposition satisfying the inviscid criterion. In other words, we did not prove that it is necessary. We only showed that the Favre decomposition is sufficient to satisfy the inviscid criterion.

As we mentioned in the introduction, while our equations (9),(11),(12) coincide to a considerable extent with those that are employed in LES of compressible turbulence, their use here and in the ensuing papers will be for rather different purposes. Whereas the primary goal in LES is to model the subgrid terms, the aim here is to develop a physical understanding of these terms and estimate their contributions at different scales, including limits of small $\ell \ll L$, through exact mathematical analysis.

4 Inertial dynamics

Now that we have isolated viscous effects to the smallest scales ℓ_μ, where ℓ_μ is defined as the scale at which viscous effects become significant in kinetic energy balance (12), we can study the dynamics at scales $\ell \gg \ell_\mu$.

4.1 Deformation work

The first term in kinetic energy SGS flux, Π_ℓ in eq. (12), is the contribution from *deformation work* done by *large-scale strain* $\partial_j\widetilde{u}_i$ against the *subgrid stress* $\overline{\rho}\,\widetilde{\tau}(u_i,u_j)$ (see for example [22]). This is similar to its incompressible counterpart except that the strain is not traceless here. It acts as a sink in the large-scale kinetic energy budget (12) and represents that part of the kinetic energy transferred from scales larger than ℓ to smaller scales at point $\mathbf{x}$ in the flow.

Furthermore, $\Pi_\ell(\mathbf{x})$ is Galilean invariant due to the subtracted large-scale terms in definition (10) of the turbulent stress. Other definitions of the SGS flux are possible such as $\widetilde{u}_i\partial_j(\overline{\rho}\widetilde{\tau}(u_i,u_j))$ which differs from our definition (13) by a total gradient $\partial_j(\overline{\rho}\widetilde{u}_i\widetilde{\tau}(u_i,u_j))$. However, this definition is not pointwise Galilean invariant, so the amount of "energy cascade" at any point $\mathbf{x}$ in the fluid according to such a definition would depend on the observer's velocity! [11], [18], and [10] all emphasized the importance of Galilean invariance. More recently, [6] and [2] showed that Galilean invariance was necessary for scale-locality of the cascade. There are non-Galilean-invariant terms in our budget (12) but, as is physically natural, they are all associated with space transport $\mathbf{J}$ of kinetic energy.

Another physical requirement on the flux $\Pi_\ell(\mathbf{x})$ is that it should vanish in the absence of fluctuations at scales smaller than ℓ (or a moderate fraction thereof); for example, when ℓ is equal to $2\pi/K_{max}$, where K_{max} is the maximum wavenumber in a direct numerical simulation (DNS). This is satisfied by our definition of $\Pi_\ell(\mathbf{x})$ identically at every point $\mathbf{x}$ in the flow. Other definitions of an energy flux are possible, such as the "unsubtracted flux" of an incompressible flow critiqued in [6] and [2],

$$\Pi_\ell^{uns}(\mathbf{x}) \equiv \overline{u}_i\,\overline{\mathcal{NL}}_i, \qquad \mathcal{NL}_i = \partial_j(\rho u_i u_j)$$

which is often employed in literature that considers the sharp-spectral filter. Here, $\mathcal{NL}_i$ denotes the nonlinearity in the momentum equation.

Using such a filter, the "unsubtracted flux" across wavenumber K is computed as

$$\Pi_K^{uns}(\mathbf{x}) = \sum_{|\mathbf{p}|<K}\hat{u}_i(\mathbf{p})e^{i\mathbf{p}\cdot\mathbf{x}}\sum_{|\mathbf{k}|<K}\widehat{\mathcal{NL}}_i(\mathbf{k})e^{i\mathbf{k}\cdot\mathbf{x}}.$$

Taking $K = K_{max}$, we have $\Pi_{K_{max}}^{uns}(\mathbf{x}) = u_i\partial_j(\rho u_i u_j)$ which is in general nonzero. It is only after averaging over all space (or in the absence of flow beyond the domain boundary) that one gets

$$\langle\Pi_{K_{max}}^{uns}\rangle = \langle\partial_j(\rho\frac{|\mathbf{u}|^2}{2}u_j)\rangle = \sum_{|\mathbf{k}|<K_{max}}\hat{u}_i^*(\mathbf{k})\,\widehat{\mathcal{NL}}_i(\mathbf{k}) = 0.$$

Similar considerations apply for compressible flows, where an unsubtracted flux may be defined as

$$\Pi_\ell^{uns}(\mathbf{x}) \equiv \widetilde{u}_i\,\overline{\mathcal{NL}}_i - \frac{1}{2}|\widetilde{\mathbf{u}}|^2\,\overline{\mathcal{N}}, \qquad \mathcal{NL}_i = \partial_j(\rho u_i u_j), \qquad \mathcal{N} = \partial_j(\rho u_j),$$

with $\mathcal{NL}_i$ and $\mathcal{N}$ denoting the nonlinearities in momentum and density equations, respectively. This flux does not vanish in general when $\ell = K_{max}^{-1}$, except after space-averaging.

4.2 Baropycnal work

The other part of kinetic energy flux, Λ_ℓ in eq. (12), is intrinsically due to compressibility effects and vanishes in the absence of density variations. It represents work done by a *large-scale pressure-gradient force*[2] $\overline{\rho}^{-1}\boldsymbol{\nabla}\overline{P}$ against *subscale mass flux*[3] $\overline{\tau}(\rho, \mathbf{u})$. We shall refer to $\Lambda_\ell(\mathbf{x})$ as *baropycnal work* due to its inherent dependence on pressure and density variations. It is not entirely due to baroclinic effects for it can be non-zero even when small-scale density variations and $\boldsymbol{\nabla}\overline{P}_\ell$ are aligned as shown in Figure 1. Similar to Π_ℓ, it also acts as a sink in the large-scale kinetic energy budget (12), is pointwise Galilean invariant, and vanishes identically at every $\mathbf{x}$ in the absence of fluctuations at scales $< \ell$. Baropycnal work is known to play a major role in turbulent combustion (see for example [21] and [12]). It has also been recently observed by [13] and [14] to play a major role in turbulence production in buoyancy-driven flows with significant density differences. In such flows, termed "variable-density flows" by [13], there are two or more incompressible miscible fluid species, such as water and brine, which have significantly different densities.

The physical mechanism behind this part of the SGS flux, illustrated in Figure 1, is simple. In a frame *co-moving* with a ball of radius ℓ in the flow, a pressure-gradient force $\overline{\rho}^{-1}\boldsymbol{\nabla}\overline{P}_\ell$ at scales $> \ell$ acts on the ball within which the fluid has non-uniform density (density variations at scales $< \ell$). Per unit time, if $\rho_2 > \rho_1$ in Figure 1 and $|\mathbf{u}_1| = |\mathbf{u}_2| = u$, then parcel 2 gains $\overline{\rho}^{-1}\boldsymbol{\nabla}\overline{P}_\ell\rho_2 u$ in kinetic energy from the large-scales and parcel 1 loses $\overline{\rho}^{-1}\boldsymbol{\nabla}\overline{P}_\ell\rho_1 u$ to the large-scales. On aggregate, scales $< \ell$ in this ball would gain $\overline{\rho}^{-1}\boldsymbol{\nabla}\overline{P}_\ell(\rho_2 - \rho_1)u$ in kinetic energy from the large-scales. This effect would vanish in the absence of density variations. Note that if $|\mathbf{u}_1|$ does not equal $|\mathbf{u}_2|$, then the whole ball will have an average large-scale velocity $\overline{\mathbf{u}}_\ell = \mathbf{u}_1 + \mathbf{u}_2$ which does not play a role in such a process of inter-scale energy transfer —hence the relevance of the premise of a co-moving frame.

Similar to deformation work, $\Pi_\ell(\mathbf{x})$, baropycnal work, $\Lambda_\ell(\mathbf{x})$, is not sign-definite in space. We expect that at points $\mathbf{x}$ in the flow where large-scale pressure gradient $\boldsymbol{\nabla}\overline{P}_\ell(\mathbf{x})$ opposes the density gradient due to small-scale fluctuations ($\rho_2 > \rho_1$ in Figure 1), then a motion similar to that of a Rayleigh-Taylor instability would ensue such that velocities $\mathbf{u}_1$ and $\mathbf{u}_2$ will be as illustrated in Figure 1. We, therefore, expect that at such points in the flow $\Lambda_\ell(\mathbf{x})$ will be positive and transfer energy to small scales. Whether, on average, the subscale mass flux $\overline{\tau}(\rho, \mathbf{u})$ would correlate positively or negatively with large-scale acceleration field $\overline{\rho}^{-1}\boldsymbol{\nabla}\overline{P}$ can be determined empirically.

Despite the recognition of Λ_ℓ's importance in turbulent combustion and variable-density flows, to the best of our knowledge, the presence of baropycnal work has not been appreciated in literature on LES of compressible turbulence. In the LES community, this term has always been lumped with $\overline{P}_\ell\boldsymbol{\nabla}\boldsymbol{\cdot}\overline{\mathbf{u}}_\ell$ in the

[2]The term"pressure-gradient force" is used in the meteorology literature. It is not a force but an acceleration.
[3]Here, "flux" denotes a flux in space, not in scale.

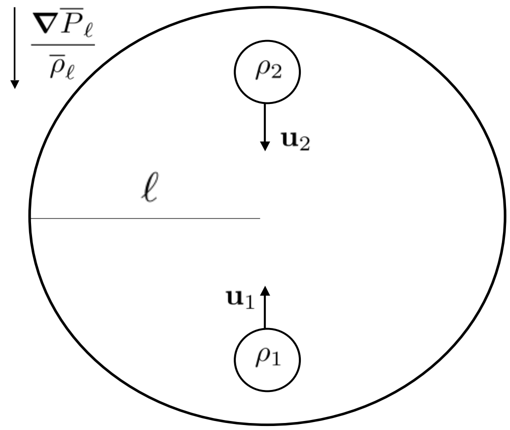

Figure 1: Heuristic explanation of the physics behind baropycnal work, Λ_ℓ. In a ball of radius ℓ, there are two small-scale fluid parcels of equal size with densities ρ_1 and ρ_2, and antiparallel velocities $\mathbf{u}_1$ and $\mathbf{u}_2$, respectively. In a frame co-moving with the ball, we must have $|\mathbf{u}_1| = |\mathbf{u}_2|$.

form $\overline{P}_\ell \nabla\cdot\widetilde{\mathbf{u}}_\ell$ (plus an additional space-transport term) and treated as a large-scale pressure dilatation which does not require modelling. We do not make any claim about the importance of baropycnal work relative to deformation work, which we believe is an issue that deserves careful empirical investigation. The reason we keep baropycnal work separate from pressure dilatation is due to a fundamental distinction between the two. Notice that both deformation work, Π_ℓ, and baropycnal work, Λ_ℓ, involve *large-scale* fields acting against *small-scale* fluctuations. This makes them capable of transferring energy *across* scales. On the other hand, pressure dilatation, $\overline{P}_\ell \nabla\cdot\overline{\mathbf{u}}_\ell$, involves only large-scales and cannot transfer energy directly across scales.

4.3 Pressure dilatation

The *pressure dilatation* $-\overline{P}\nabla\cdot\overline{\mathbf{u}}$ in eq. (12) represents conversion of large-scale kinetic energy to internal energy through compression. In the incompressible limit, this vanishes at every point $\mathbf{x}$. Unlike Π_ℓ and Λ_ℓ, pressure dilatation only contains scales $> \ell$ (at least for filter kernels $\hat{G}(\mathbf{k})$ compact in Fourier space). Therefore, it is not involved in the transfer of energy *across* scale ℓ and it does not vanish in the absence of subscale fluctuations.

5 Kinetic energy injection

Under statistically steady-state conditions and after space averaging, the large-scale kinetic energy budget (12) at scales $\ell \gg \ell_\mu$ becomes

$$\langle \Pi_\ell \rangle + \langle \Lambda_\ell \rangle - \langle \overline{P}_\ell \nabla\cdot\overline{\mathbf{u}}_\ell \rangle = \langle \epsilon_\ell^{inj} \rangle. \tag{20}$$

We assumed that none of the kinetic energy is transported beyond the domain boundary. We also dropped viscous dissipation, $\langle D_\ell \rangle$, which we have proved to be negligible. If, furthermore, the injection of kinetic energy can be made localized to the very largest scales $L \gg \ell$, then one should be able to study non-linear inertial dynamics at intermediate scales $L \gg \ell \gg \ell_\mu$ over which the left-hand side of eq. (20) becomes a constant, independent of ℓ.

The injection can be made localized to small wavenumbers if the external acceleration field $\mathbf{F}(\mathbf{x})$ has only small Fourier wavenumbers. To show this, we will denote a field in a periodic domain $[0, 2\pi)^3$, coarse-grained with the sharp-spectral filter to retain only Fourier modes $|\mathbf{k}| < K$, by

$$\mathbf{a}^{<K}(\mathbf{x}) \equiv \sum_{|\mathbf{k}| \leq K} d\mathbf{k}\ \hat{\mathbf{a}}(\mathbf{k}) e^{i\mathbf{k}\cdot\mathbf{x}}. \tag{21}$$

This is similar to $\overline{\mathbf{a}}_\ell(\mathbf{x})$ with ℓ being of the same order as K^{-1}. We will also employ in our argument the notation $[K_1, K_2]$ for a band of Fourier modes $K_1 < |\mathbf{k}| \leq K_2$ and denote the corresponding *band-pass filtered field* by $\mathbf{a}^{[K_1,K_2]}$. Now consider an imposed acceleration $\mathbf{F}^{<K_0}$ with modes restricted to small wavenumbers $\leq K_0 \ll K$. The net energy injection from (17) is, thus,

$$\langle \epsilon_K^{inj} \rangle = \left\langle \frac{(\rho u_i)^{<K}}{\rho^{<K}} \left(\rho F_i^{<K_0}\right)^{<K} \right\rangle. \tag{22}$$

We have the following exact identity:

$$\begin{aligned} \left(\rho F^{<K_0}\right)^{<K}(\mathbf{x}) &= \left(\rho^{<K+K_0} F^{<K_0}\right)^{<K} = \rho^{<K-K_0} F^{<K_0} + \left(\rho^{[K-K_0,K+K_0]} F^{<K_0}\right)^{<K} \\ &= \rho^{<K} F^{<K_0} - \rho^{[K-K_0,K]} F^{<K_0} + \left(\rho^{[K-K_0,K+K_0]} F^{<K_0}\right)^{<K}. \end{aligned} \tag{23}$$

The first equality follows from the fact that a Fourier mode $\hat{\rho}(\mathbf{p})$ with $|\mathbf{p}| > K + K_0$ along with $\hat{\mathbf{F}}(\mathbf{q})$ with $|\mathbf{q}| < K_0$ cannot contribute to the product $\rho\mathbf{F}$ with Fourier modes $\mathbf{k} = \mathbf{p} + \mathbf{q}$ restricted to $|\mathbf{k}| < K$. The second equality is due to $(\rho^{<K-K_0} F^{<K_0})^{<K} = \rho^{<K-K_0} F^{<K_0}$. Notice that in the second and third terms of the last expression in (23), density has only modes in a thin band of width at most $2K_0$ around K. As $K_0/K \to 0$, a factor $\rho^{[K-K_0,K+K_0]}$ should become negligible due to a decaying density spectrum and, therefore, the second and third terms in (23) vanish. Using this observation in (22) implies that the net energy injection with $K_0 \ll K$ reduces to

$$\langle \epsilon^{inj} \rangle \approx \left\langle \frac{(\rho u_i)^{<K}}{\rho^{<K}} \rho^{<K} F^{<K_0} \right\rangle = \left\langle\, (\rho u_i)^{<K_0} F^{<K_0} \right\rangle, \tag{24}$$

where the last equality follows from orthogonality of Fourier modes. In Appendix A, we present a rigorous proof of the above statement under precise (and weak) conditions. Our result (24) implies the non-trivial possibility to make injection localized to wavenumbers $< K_0$ by employing an external acceleration field band-limited to wavenumbers $< K_0$.

In arriving at eq. (24), we had to stir with an *external acceleration* field such that the force in (2) is weighted by instantaneous density $\rho(\mathbf{x}, t)$. Had we stirred the momentum equation with an *external force* $\boldsymbol{\mathcal{F}}^{<K_0}$ instead of $\rho\mathbf{F}^{<K_0}$, the injection would have had the form $\langle\epsilon^{inj}\rangle = \left\langle\frac{(\rho u_i)^{<K}}{\rho^{<K}}\mathcal{F}_i^{<K_0}\right\rangle \approx \left\langle\mathbf{u}^{<K_0}\boldsymbol{\cdot}\boldsymbol{\mathcal{F}}^{<K_0}\right\rangle$, for $K_0 \ll K$. We shall not present a detailed proof of the latter statement but the idea d details behind it are similar to those leading to (24). Finally, we note that the acceleration field $\mathbf{F}^{<K_0}$ in momentum eq. (2) is of a general form and can have both solenoidal and irrotational components.

Under such conditions, one obtains that the sum of kinetic energy flux and pressure dilatation,

$$\langle\Pi_\ell\rangle + \langle\Lambda_\ell\rangle - \langle\overline{P}_\ell\boldsymbol{\nabla}\boldsymbol{\cdot}\overline{\mathbf{u}}_\ell\rangle = (\text{const.}) \tag{25}$$

in (20), is constant, independent of wavenumber $K = \ell^{-1}$. This in itself is not an analogue to Kolmogorov's 4/5-th law since it contains the pressure dilatation term which does not involve energy transfer across scales. However, we shall show in a subsequent paper that it is possible to derive a result analogous to the 4/5-th law under a reasonable assumption on the co-spectrum of pressure dilatation.

5.1 Special case: Rayleigh-Taylor flow

We shall apply our result on localized injection to the case of a Rayleigh-Taylor flow driven by gravitational forces. We ask the following question: At what scales does mean conversion of gravitational potential energy to kinetic energy take place?

We shall show that, in the presence of a spatially uniform gravitational field $\mathbf{g}(\mathbf{x}) = \mathbf{g}$, such conversion into kinetic energy only takes place at the largest scale —that of the domain size. Consider the large-scale kinetic energy equation (12) where $\mathbf{F}(\mathbf{x})$ is replaced with $\mathbf{g}$ in ϵ_ℓ^{inj}. We have from (17) that net input of kinetic energy due to gravity is

$$\langle\epsilon_\ell^{inj}\rangle = \langle\widetilde{\mathbf{u}}_\ell\boldsymbol{\cdot}\overline{\rho}_\ell\widetilde{\mathbf{g}}_\ell\rangle = \langle\overline{\rho\mathbf{u}}_\ell\boldsymbol{\cdot}\mathbf{g}\rangle = \mathbf{g}\boldsymbol{\cdot}\langle\rho\mathbf{u}\rangle \tag{26}$$

where the last equality follows from $\int d\mathbf{x}\overline{f}_\ell(\mathbf{x}) = \int d\mathbf{r}G_\ell(\mathbf{r})\int d\mathbf{x}f(\mathbf{x}+\mathbf{r}) = \langle f\rangle$. Result (26) shows that mean injection is independent of scale ℓ and only takes place at the scale of the domain, L_{dom}.

Put in more detail, the mean kinetic energy $\langle\overline{\rho}_\ell|\widetilde{\mathbf{u}}_\ell|^2\rangle/2$ at scales $> \ell$ increases at a rate $\langle\widetilde{\mathbf{u}}_\ell\boldsymbol{\cdot}\overline{\rho}_\ell\widetilde{\mathbf{g}}_\ell\rangle$ due to gravitational forces. Consider a sequence of scales $\ell_1 > \ell_2 > \cdots > \ell_n > \ldots$ and the average amount of potential energy being converted into mean kinetic energy, $\langle\epsilon_{\ell_n}^{inj}\rangle \equiv \langle\widetilde{\mathbf{u}}_{\ell_n}\boldsymbol{\cdot}\overline{\rho}_{\ell_n}\widetilde{\mathbf{g}}_{\ell_n}\rangle$, at successively larger sets of scales $[\ell_n, L_{dom})$. In general, as $n \to \infty$, $\langle\epsilon_{\ell_n}^{inj}\rangle$ approaches the total amount of potential energy converted into kinetic form. The fact that $\langle\epsilon_{\ell_n}^{inj}\rangle$ is independent of ℓ_n over the entire scale-range $[0, L_{dom})$ implies that all conversion takes place at the domain scale.

Our argument demonstrates the power of the filtering approach[4] in analyzing non-linear scale inter-

[4] In this particular case, with a constant $\mathbf{g}$ containing only a $k = 0$ mode, the same conclusion would have also been possible by examining the injection into mean subscale kinetic energy, $\langle\overline{\rho|\mathbf{u}|^2}_\ell - \overline{\rho}_\ell|\widetilde{\mathbf{u}}_\ell|^2\rangle/2$, which may be easily determined to equal zero at any ℓ.

actions. We were able to arrive at our answer precisely because the filtering technique allows for probing a *continuous range of scales*, in contrast to the traditional averaging approach.

The conclusion is probably non-intuitive at face value because, in a Rayleigh-Taylor flow, “fingers” of heavy fluid *at any scale* penetrate the lighter fluid as they descend, converting potential to kinetic energy. It seems to contradict our result that conversion only takes place at the domain scale. Key to understanding such an ostensible paradox is the definition of kinetic energy based on a Favre scale-decomposition. Mean kinetic energy at scale ℓ may be rewritten as $\langle|\overline{\rho\mathbf{u}}_\ell|^2/\overline{\rho}_\ell\rangle/2$, which emphasizes the central role of *momentum* in defining scale. Indeed, result (26) demonstrates that it is only the $k=0$ mode of momentum which participates in converting potential energy into kinetic energy at scale with mode $k=0$, $\langle\rho\mathbf{u}\rangle^2/\langle\rho\rangle/2$. While momentum at scale ℓ can have contributions from density and velocity at *all* scales —for example, mean momentum $\langle\rho\mathbf{u}\rangle = \sum_{\mathbf{k}}\hat{\rho}(\mathbf{k})\hat{\mathbf{u}}(-\mathbf{k})$ — the scale ℓ of kinetic energy depends on that of momentum $\overline{\rho\mathbf{u}}_\ell$ and not on, for instance, $\overline{\rho}_\ell\overline{\mathbf{u}}_\ell$.

6 Compressibility effects in the SGS flux

Our scale-decomposition allowed us to identify two SGS sinks for the large-scale kinetic energy budget, namely deformation work, Π_ℓ, and baropycnal work, Λ_ℓ.

While Π_ℓ also represents a cascade mechanism in incompressible turbulence, Λ_ℓ is solely due to density variations and thus is intrinsic to compressible flows. However, even deformation work has contributions from compressibility effects. We shall show in a subsequent paper that $\Pi_\ell = \overline{\rho}\partial_j\overline{u}_i\overline{\tau}(u_i,u_j) + \ldots 8 \text{ terms} \ldots$, using exact identities. If we consider that part of Π_ℓ equal to $\overline{\rho}\partial_j\overline{u}_i\overline{\tau}(u_i,u_j)$, we see that compressive modes with $\mathbf{k}\cdot\hat{\mathbf{u}}(\mathbf{k}) \neq 0$ can play an important role in deformation work. This is because the large-scale strain $\boldsymbol{\nabla}\overline{\mathbf{u}}$ is not traceless. The effect of compressive modes is best illustrated by the one-dimensional Burger’s flow,

$$\partial_t u + \partial_x(\frac{u^2}{2}) = \nu\partial_{xx}u. \tag{27}$$

A large-scale kinetic energy budget analogous to (12) can be derived (see for example [5]):

$$\partial_t\frac{|\overline{u}|^2}{2} + \partial_x\left\{\frac{\overline{u}^3}{3} + \overline{u}\;\overline{\tau}(u,u) - \nu\partial_x\frac{|\overline{u}|^2}{2}\right\} = -\Pi^{burg} - \nu|\partial_x\overline{u}|^2, \tag{28}$$

in which viscous dissipation on the right-hand-side is negligible and

$$\Pi^{burg} \equiv -\partial_x\overline{u}\;\overline{\tau}(u,u) \tag{29}$$

is the only sink. There are no shearing motions in flow (27). The only way energy cascades to small scales is through Π^{burg} which is solely due to compressive modes. As is well-known, this cascade is manifested in the formation of shocks.

In general, a simple measure that quantifies kinematic role of compressibility on deformation work (13) is

$$\Pi_\ell^{comp} \equiv \Pi_\ell + \overline{\rho}\ \partial_j \overline{u^s}_i\ \overline{\tau}(u_i^s, u_j^s), \tag{30}$$

where the velocity $\mathbf{u} = \mathbf{u}^s + \mathbf{u}^c$ is decomposed into solenoidal and irrotational components, $\mathbf{u}^s$ and $\mathbf{u}^c$, respectively.

7 Summary

In this paper we have shown that viscous diffusion and dissipation can be isolated to the smallest scales in a compressible flow by using a proper scale-decomposition. Guided by this physical requirement, which we call the "inviscid criterion", we found that a Favre decomposition of the momentum and kinetic energy into large-scale and small-scale components is sufficient to guarantee a negligible role of molecular viscosity in the large-scale dynamics of high Reynolds number flows.

We were also able to establish through an exact analysis that mean kinetic energy injection can also be made localized to the largest scales in a flow by proper stirring. We discussed the special case of buoyancy-driven flows in which stirring is due to an external spatially-uniform acceleration field, and showed that mean injection of kinetic energy occurs only at the very largest scale in the system.

Localizing viscous dissipation to the smallest scales and energy injection to the largest scales is necessary to allow for studying inertial dynamics at intermediate scales. Under steady-state conditions, satisfying these two ingredients implies that the sum of mean SGS kinetic energy flux and pressure dilatation, $\langle \Pi_\ell + \Lambda_\ell \rangle - \langle \boldsymbol{\nabla}\overline{P}_\ell {\cdot} \overline{\mathbf{u}}_\ell \rangle$, is constant, independent of scale ℓ over the intermediate range $L \gg \ell \gg \ell_\mu$. This is simply a consequence of energy conservation; whatever energy is input into the system has to either reach dissipation scales by way of the SGS flux or get converted into internal energy through pressure dilatation.

The results of this paper lay the groundwork for ensuing work to tackle basic issues concerning the physics of compressible turbulence. Questions on whether energy reaches dissipation scales through a local cascade process, on whether an analogue to Kolmogorov's 4/5-th law exists, and on how spectra and structure functions should scale in compressible turbulence, will form the subject of subsequent works which build on the conclusions of this paper.

Acknowledgements. I am indebted to G. L. Eyink for invaluable discussions on turbulence over many years. I want to especially thank D. Livescu for his input on compressible and variable-density flows. I also thank S.

S. Girimaji for helpful suggestions and X. Asay-Davis for constructive comments on an earlier version of the manuscript. I am appreciative of S. Kurien, S. Li, and H. Li for their encouragement and support during this project. This research was performed under the auspices of the U.S. Department of Energy at LANL under Contract No. DE-AC52-06NA25396 and supported by the LANL/LDRD program.

A Viscous effects

The following proposition shows that viscous terms in the balance (9) of large-scale momentum $\overline{\rho}_\ell \widetilde{\mathbf{u}}_\ell$ are negligible when $\mu/\ell^2 \ll 1$. The proof is identical to that given in [5] and [1].

Proposition 1. *If velocity solutions $\mathbf{u}^\mu$ of the compressible Navier-Stokes equation (1)-(3) over a domain Ω have finite 2nd-order moments: $\int_\Omega d\mathbf{x}|\mathbf{u}^\mu|^2 < \infty$, then viscous terms in the large-scale momentum eq. (7) vanish pointwise as $\mu \to 0$.*

Proof of Proposition 1: The viscous terms in eq. (7) can be bounded using Cauchy-Schwartz inequality,

$$\begin{aligned}\left|\mu\nabla^2\overline{\mathbf{u}^\mu}_\ell(\mathbf{x})\right| &= \left|\frac{\mu}{\ell^2}\int d\mathbf{r}(\nabla^2 G)_\ell(\mathbf{r})\mathbf{u}^\mu(\mathbf{x}+\mathbf{r})\right| \\ &\leq \frac{\mu}{\ell^2}\big\|(\nabla^2 G)_\ell\big\|_2\big\|\mathbf{u}^\mu(\mathbf{r})\big\|_2,\end{aligned}$$

with $\|\ldots\|_p = \langle|\ldots|^p\rangle^{1/p}$ the L_p-norm. Since $G(\mathbf{r}) \in C^\infty$, its derivatives are (uniformly) bounded and for any fixed $\ell > 0$, $\big\|(\nabla^2 G)_\ell\big\|_2 <$ (const.). Hence, the viscous term $\mu\nabla^2\overline{\mathbf{u}^\mu}_\ell \to 0$ at every point in space in the limit of vanishing viscosity. The same argument applies to $\mu\boldsymbol{\nabla}\boldsymbol{\nabla}\boldsymbol{\cdot}\overline{\mathbf{u}^\mu}$. □

The following proposition proves that viscous dissipation of large-scale kinetic energy $\frac{1}{2}\overline{\rho}_\ell|\widetilde{\mathbf{u}}_\ell|^2$ is negligible when $\mu/\ell^2 \ll 1$.

Proposition 2. *If solutions $(\rho^\mu, \mathbf{u}^\mu)$ of the compressible Navier-Stokes equation (1)-(3) over a bounded domain Ω have finite 3rd-order moments: $\int_\Omega d\mathbf{x}|\rho^\mu|^3 < \infty$ and $\int_\Omega d\mathbf{x}|\mathbf{u}^\mu|^3 < \infty$, then viscous terms in the large-scale kinetic energy budget (12) vanish pointwise as $\mu \to 0$.*

Proof of Proposition 2: In the following, we shall drop superscript μ from $(\rho^\mu, \mathbf{u}^\mu)$. Using the exact identity $\widetilde{\mathbf{u}} = \overline{\mathbf{u}} + \overline{\tau}(\rho, \mathbf{u})/\overline{\rho}$, we have

$$\big|\partial_j\widetilde{u}_i(\mathbf{x})\big| \leq \big|\partial_j\overline{u}_i\big| + \big|\frac{1}{\overline{\rho}}\partial_j\overline{\tau}(\rho, u_i)\big| + \big|\frac{1}{\overline{\rho}^2}\overline{\tau}(\rho, u_i)\partial_j\overline{\rho}\big|. \tag{31}$$

The first term $\big|\partial_j\overline{u}_i\big|$ is bounded by ℓ^{-1}(const.)$\|\boldsymbol{\nabla}G_\ell\|_2\|\mathbf{u}\|_3$ through an argument identical to that in Proposition 1 and using the fact that $\|\mathbf{u}\|_2 \leq$ (const.)$\|\mathbf{u}\|_3$ over a bounded domain.

The second term in (31) can be rewritten as

$$\frac{1}{\overline{\rho}}\partial_j\overline{\tau}(\rho,u_i) = -\frac{1}{\overline{\rho}\ell}\Big[\int d\mathbf{r}(\partial_j G)_\ell(\mathbf{r})\rho(\mathbf{x}+\mathbf{r})u_i(\mathbf{x}+\mathbf{r})$$

$$-\int d\mathbf{r}(\partial_j G)_\ell(\mathbf{r})\rho(\mathbf{x}+\mathbf{r})\int d\mathbf{r}' G_\ell(\mathbf{r}')u_i(\mathbf{x}+\mathbf{r}') - \int d\mathbf{r} G_\ell(\mathbf{r})\rho(\mathbf{x}+\mathbf{r})\int d\mathbf{r}'(\partial_j G)_\ell(\mathbf{r}')u_i(\mathbf{x}+\mathbf{r}')\Big],$$

using integration by parts. Employing the 3-3-3 Hölder's inequality, this expression is bounded by

$$\ell^{-1}\frac{1}{\overline{\rho}}\big\|(\nabla G)_\ell\big\|_3(1+2\|1\|_3^2\|G_\ell\|_3)\|\rho\|_3\|\mathbf{u}\|_3$$

The third term in (31) can be rewritten as

$$\frac{1}{\overline{\rho}^2}\overline{\tau}(\rho,u_i)\partial_j\overline{\rho} = -\overline{\rho}^{-2}\ell^{-1}\int d\mathbf{r}(\partial_j G)_\ell(\mathbf{r})\rho(\mathbf{x}+\mathbf{r})$$

$$\times\Big[\int d\mathbf{r} G_\ell(\mathbf{r})\rho(\mathbf{x}+\mathbf{r})u_i(\mathbf{x}+\mathbf{r}) - \int d\mathbf{r} G_\ell(\mathbf{r})\rho(\mathbf{x}+\mathbf{r})\int d\mathbf{r}' G_\ell(\mathbf{r}')u_i(\mathbf{x}+\mathbf{r}')\Big].$$

Using Hölder's inequality, this is bounded by

$$\ell^{-1}\frac{1}{\overline{\rho}^2}\big\|(\nabla G)_\ell\big\|_3\|1\|_3\|G_\ell\|_3(1+\|1\|_3\|G_\ell\|_3)\|\rho\|_3^2\|\mathbf{u}\|_3$$

Finally, the viscous term is bounded by

$$\mu\big|\partial_j\widetilde{u}_i\partial_j\overline{u}_i\big| \le \frac{\mu}{\ell^2}\|\mathbf{u}\|_3^2\Big[(\text{const.}) + (\text{const.})\frac{\|\rho\|_3}{\overline{\rho}} + (\text{const.})\frac{\|\rho\|_3^2}{\overline{\rho}^2}\Big].$$

The coarse-grained density $\overline{\rho}_\ell(\mathbf{x})$, for positive filter kernels $G(\mathbf{r}) \ge 0$, is proportional to the mass in a ball of radius ℓ centered around $\mathbf{x}$. At any point in the flow, this is bounded away from zero. Hence, for any fixed $\ell > 0$, the viscous term $\mu\partial_j\widetilde{u}_i\partial_j\overline{u}_i \to 0$ at every point in space in the limit of vanishing viscosity. The same argument applies to $\mu\partial_j\widetilde{u}_j\partial_i\overline{u}_i$. □

B Kinetic energy injection

The following proposition proves that mean kinetic energy injection is localized to the largest scales $\gtrsim L$. In the text, we presented an informal argument using the sharp-spectral filter. However, this filter's kernel is not positive in physical space, which prevents us from inferring that $(\rho^{<K})^{-1}(\mathbf{x})$ is bounded at every point $\mathbf{x}$ [5]. For this reason, we will use in our proof below the Fejér summation of Fourier series. This is defined as

$$\mathbf{a}^{(K)}(\mathbf{x}) \equiv \frac{1}{K}\sum_{Q=1}^{K}\mathbf{a}^{<K}(\mathbf{x}), \tag{32}$$

[5] We are not claiming that coarse-graining with a sharp-spectral filter yields non-localized injection, but only stating that we cannot prove otherwise.

where $\mathbf{a}^{<K}(\mathbf{x})$ is coarse-grained using the sharp-spectral filter as defined in (21). The Fejér kernel $\Gamma_K(\mathbf{r})$ is positive in physical space and compactly supported in a ball of radius K in Fourier space.

To show that mean kinetic energy injection is localized to the largest scales $\gtrsim K_0^{-1}$, we need to prove that

$$\langle \epsilon_K^{inj} \rangle = \left\langle \frac{(\rho u_i)^{(K)}}{\rho^{(K)}} (\rho F_i^{<K_0})^{(K)} \right\rangle \longrightarrow \left\langle \Big((\rho u_i)^{(K)} \Big)^{<K_0} F_i^{<K_0} \right\rangle, \tag{33}$$

as $K_0/K \to 0$.

To this end, we will need an identity similar to (23) in the text for the Fejér filtered quantity. For $K \geq 2K_0 \geq 2$ (the special case of $K \geq 2K_0 = 0$ is straightforward and was discussed in § 5.1), we have:

$$(\rho F^{<K_0})^{(K)} = \frac{1}{K} \sum_{Q=2K_0}^{K} (\rho F^{<K_0})^{<Q} + \frac{1}{K} \sum_{Q=1}^{2K_0-1} (\rho F^{<K_0})^{<Q}$$

Using identity (23), the first term on the right hand side can be rewritten as

$$\frac{1}{K} \sum_{Q=2K_0}^{K} (\rho F^{<K_0})^{<Q} = \frac{1}{K} \sum_{Q=2K_0}^{K} \rho^{<Q} F^{<K_0} - \rho^{[Q-K_0,Q]} F^{<K_0} + (\rho^{[Q-K_0,Q+K_0]} F^{<K_0})^{<Q}$$

$$= \rho^{(K)} F^{<K_0} - \frac{1}{K} \sum_{Q=1}^{2K_0-1} \rho^{<Q} F^{<K_0} + \frac{1}{K} \sum_{Q=2K_0}^{K} -\rho^{[Q-K_0,Q]} F^{<K_0} + (\rho^{[Q-K_0,Q+K_0]} F^{<K_0})^{<Q}.$$

We finally have the identity

$$\begin{aligned} (\rho F^{<K_0})^{(K)} &= \rho^{(K)} F^{<K_0} \\ &+ \left\{ \frac{1}{K} \sum_{Q=1}^{2K_0-1} (\rho F^{<K_0})^{<Q} - \rho^{<Q} F^{<K_0} \right. \\ &+ \left. \frac{1}{K} \sum_{Q=2K_0}^{K} -\rho^{[Q-K_0,Q]} F^{<K_0} + (\rho^{[Q-K_0,Q+K_0]} F^{<K_0})^{<Q} \right\} \end{aligned} \tag{34}$$

Proposition 3. *If solutions $(\rho, \mathbf{u})$ of the compressible Navier-Stokes equation (1)-(3) over a periodic domain $\mathbb{T} = [0, 2\pi)^3$ satisfy $\rho_{rms}^2 \equiv \int_{\mathbb{T}} d\mathbf{x} |\rho|^2 < \infty$ and $(\rho \mathbf{u})_{rms}^2 \equiv \int_{\mathbb{T}} d\mathbf{x} |\rho \mathbf{u}|^2 < \infty$, then*

$$\begin{aligned} &\lim_{K_0/K \to 0} \left| \langle \epsilon_K^{inj} - \Big((\rho u_i)^{(K)} \Big)^{<K_0} F_i^{<K_0} \rangle \right| \\ \leq\ & \rho_{rms}\, (\rho \mathbf{u})_{rms} \lim_{K_0/K \to 0} \left\{ (\text{const.}) \sqrt{\frac{K_0}{K}} + (\text{const.}) \frac{K_0}{K} \right\} = 0. \end{aligned} \tag{35}$$

Proof of Proposition 3:

We first note that $\rho_{rms} < \infty$ implies that the density spectrum, $E^\rho(k) \equiv \sum_{k-0.5<|\mathbf{k}|<k+0.5} |\hat{\rho}(\mathbf{k})|^2$, is bounded by a power-law:

$$E^\rho(k) \leq (\text{const.}) \rho_{rms}^2 k^{-n}, \qquad n > 1.$$

Otherwise, ρ_{rms} increases without bound as a function of Reynolds number (see for example [8]).

Using identity (34), we have

$$\langle\epsilon_K^{inj}\rangle - \left\langle \left((\rho u_i)^{(K)}\right)^{<K_0} F_i^{<K_0}\right\rangle$$

$$= \left\langle \frac{(\rho u_i)^{(K)}}{\rho^{(K)}}\left\{\frac{1}{K}\sum_{Q=1}^{2K_0-1}(\rho F_i^{<K_0})^{<Q} - \rho^{<Q}F_i^{<K_0} + \frac{1}{K}\sum_{Q=2K_0}^{K} -\rho^{[Q-K_0,Q]}F_i^{<K_0} + (\rho^{[Q-K_0,Q+K_0]}F_i^{<K_0})^{<Q}\right\}\right\rangle$$

Applying the Hölder inequality to the right hand side, we get

$$\begin{aligned} &\left|\langle\epsilon_K^{inj}\rangle - \langle \left((\rho u_i)^{(K)}\right)^{<K_0} F_i^{<K_0}\rangle\right| \\ &\leq \left\|\tfrac{1}{\rho^{(K)}}\right\|_\infty \left\|\mathbf{F}^{<K_0}\right\|_\infty \left\|(\rho\mathbf{u})^{(K)}\right\|_2 \left\{2\tfrac{2K_0}{K}\left\|\rho\right\|_2 + \tfrac{1}{K}\textstyle\sum_{Q=2K_0}^{K} 2\left\|\rho^{[Q-K_0,Q+K_0]}\right\|_2\right\}. \end{aligned} \tag{36}$$

Since $\|\rho^{[Q-K_0,Q+K_0]}\|_2^2 = 2\int_{K-K_0}^{K+K_0} dk E^\rho(k)$, we obtain from our bound on the density spectrum that

$$\begin{aligned} \left\|\rho^{[Q-K_0,Q+K_0]}\right\|_2 &\leq (\text{const.})\rho_{rms}Q^{-(n-1)/2}\left[(1-\frac{K_0}{Q})^{-(n-1)} - (1+\frac{K_0}{Q})^{-(n-1)}\right]^{1/2} \\ &\leq (\text{const.})\rho_{rms}\sqrt{K_0}\frac{1}{Q^{n/2}}. \end{aligned}$$

This yields that

$$\begin{aligned} \frac{1}{K}\sum_{Q=2K_0}^{K}\left\|\rho^{[Q-K_0,Q+K_0]}\right\|_2 &\leq (\text{const.})\rho_{rms}\sqrt{K_0}\frac{1}{K}\sum_{Q=1}^{K}Q^{-n/2} \\ &\leq (\text{const.})\rho_{rms}\sqrt{K_0}\frac{1}{K}\int_1^K dQ\ Q^{-n/2} \\ &= \begin{cases} (\text{const.})\rho_{rms}\sqrt{\frac{K_0}{K}}\left(\frac{1}{K^{(n-1)/2}} - \frac{1}{K^{1/2}}\right) & n\neq 2 \\ (\text{const.})\rho_{rms}\sqrt{\frac{K_0}{K}}\left(\frac{\ln K}{\sqrt{K}}\right) & n=2 \end{cases} \end{aligned} \tag{37}$$

Since $\rho^{(K)}$ is proportional to the mass in a ball of radius $\sim K^{-1}$, we have $\|\left(\rho^{(K)}\right)^{-1}\|_\infty < \infty$. We can also have $\|\mathbf{F}^{<K_0}\|_\infty < \infty$ by construction, and $\|(\rho\mathbf{u})^{(K)}\|_2 \leq (\rho\mathbf{u})_{rms} < \infty$ follows from our assumptions. We finally have from (36) that

$$\begin{aligned} &\left|\langle\epsilon_K^{inj} - \left((\rho u_i)^{(K)}\right)^{<K_0} F_i^{<K_0}\rangle\right| \\ &\leq \rho_{rms}\,(\rho\mathbf{u})_{rms}\left\{(\text{const.})\sqrt{\frac{K_0}{K}} + (\text{const.})\frac{K_0}{K}\right\} \end{aligned}$$

which vanishes as $K_0/K \to 0$. □

REMARK: In our proof, we only used that the density spectrum is *bounded* by $(\text{const.})\rho_{rms}^2 k^{-n}$. This is very weak and does not even require a continuous spectrum. If the spectrum decays faster than such a bound, then $\langle\epsilon_K^{inj}\rangle \to \langle \left((\rho u_i)^{(K)}\right)^{<K_0} F_i^{<K_0}\rangle$ at a rate faster than what is stated in estimate (35).

References

[1] H. Aluie. *Hydrodynamic and Magnetohydrodynamic Turbulence: Invariants, Cascades, and Locality.* PhD thesis, The Johns Hopkins University, Baltimore, 2009.

[2] H. Aluie and G. L. Eyink. Localness of energy cascade in hydrodynamic turbulence. II. Sharp spectral filter. *Phys. Fluids*, 21(11):115108, November 2009.

[3] G. L. Eyink. Local energy flux and the refined similarity hypothesis. *J. Stat. Phys.*, 78:335–351, 1995.

[4] G. L. Eyink. Locality of turbulent cascades. *Physica D*, 207:91–116, 2005.

[5] G. L. Eyink. Course notes on turbulence theory. Available online: http://www.ams.jhu.edu/~eyink/OLD/Turbulence_ Spring08/notes.html, 2007.

[6] G. L. Eyink and H. Aluie. Localness of energy cascade in hydrodynamic turbulence. I. Smooth coarse graining. *Phys. Fluids*, 21(11):115107, November 2009.

[7] A. Favre. Statistical equations of turbulent gases. In *Problems of hydrodynamic and continuum mechanics, SIAM, Philadelphia*, pages 231–266, 1969.

[8] U. Frisch. *Turbulence. The legacy of A. N. Kolmogorov.* Cambridge University Press, UK, 1995.

[9] E. Garnier, N. Adams, and P. Sagaut. *Large Eddy Simulation for Compressible Flows.* Springer, Netherlands, 2009.

[10] M. Germano. Turbulence - The filtering approach. *J. Fluid Mech.*, 238:325–336, 1992.

[11] R. H. Kraichnan. Kolmogorov's hypotheses and Eulerian turbulence theory. *Phys. Fluids*, 7:1723–1734, 1964.

[12] P. A. Libby and K. N. C. Bray. Countergradient diffusion in premixed turbulent flames. *AIAA J.*, 19:205–213, 1981.

[13] D. Livescu and J. R. Ristorcelli. Buoyancy-driven variable-density turbulence. *J. Fluid Mech.*, 591:43–71, 2007.

[14] D. Livescu, J. R. Ristorcelli, R. A. Gore, S. H. Dean, W. H. Cabot, and A. W. Cook. High-Reynolds number Rayleigh-Taylor turbulence. *J. Turbul.*, 10:13, 2009.

[15] P. Martin, U. Piomelli, and G. V. Candler. Subgrid-scale models for compressible large-eddy simulations. *Theor. Comp. Fluid Dyn.*, 13:361–376, 2000.

[16] B. R. Pearson, T. A. Yousef, N. E. L. Haugen, A. Brandenburg, and P. A. Krogstad. Delayed correlation between turbulent energy injection and dissipation. *Phys. Rev. E*, 70(5):056301, November 2004.

[17] P. Sagaut and M. Germano. On the filtering paradigm for LES of flows with discontinuities. *J. Turbul.*, 6:23, 2005.

[18] C. G. Speziale. Galilean invariance of subgrid-scale stress models in the large-eddy simulation of turbulence. *J. Fluid Mech.*, 156:55–62, 1985.

[19] K. R. Sreenivasan. On the scaling of the turbulence energy dissipation rate. *Phys. Fluids*, 27:1048–1051, May 1984.

[20] K. R. Sreenivasan. An update on the energy dissipation rate in isotropic turbulence. *Phys. Fluids*, 10:528–529, February 1998.

[21] S. H. Starner and R. W. Bilger. LDA measurements in a turbulent diffusion flame with axial pressure. *Combust. Sci. Technol.*, 21:259–276, 1980.

[22] H. Tennekes and J. L. Lumley. *A First Course in Turbulence.* The MIT Press, Cambridge, Massachusetts, 1972.

[23] B. Vreman. *A priori test of large eddy simulation of the compressible plane mixing layer.* PhD thesis, University of Twente, Twente, 1995.

[24] B. Vreman, B. Geurts, and H. Kuerten. A priori test of large eddy simulation of the compressible plane mixing layer. *J. Eng. Math.*, 29:299–327, July 1995.